Evaluating Power of Agents from Dependence Relations in Boolean Games

Jonathan Ben-Naim
IRIT-CNRS
Toulouse University, France
bennaim@irit.fr

Emiliano Lorini
IRIT-CNRS
Toulouse University, France
lorini@irit.fr

ABSTRACT

In this paper we propose a new methodology for evaluating the relative power of agents in a strategic situation formally represented by a boolean game. The methodology consists in extracting a power ranking from the dependence relation induced by a certain boolean game. Our approach is axiomatic. We provide a number of desirable postulates that a notion of dependence is expected to satisfy and we compare competing notions of dependence, included a notion based on the concept of veto player, with respect to them. Similarly, we provide a set of postulates for power functions (i.e., the family of functions mapping dependence graphs to power rankings) and evaluate some new methods as well as existing ones (e.g., Pagerank) with respect to this set of postulates.

Keywords

Boolean games, dependence, power

1. INTRODUCTION

A *game-based power evaluation* consists in assessing the social power of the agents interacting in a given situation, formally represented by a game. It can be represented mathematically as a function that takes a game as an input and returns a ranking of the agents as output, reflecting the relative power of the agents in the game. Such an evaluation is useful, because the agents typically need to choose the peers to interact with. The agents can also represent potential partners for external users, who then will find this evaluation useful as well. Moreover, information about the relative power of the agents in a society may be useful for policy makers in order to reduce inequalities and to promote fairness. Examples of applications for game-based power evaluations include web services, recommender systems and, more generally, the design and analysis of social procedures (e.g., for resource allocation, task distribution, etc.).

The goal of the present paper is to construct and analyze game-based power evaluations whose notion of power is based on the concept of *social dependence*. The idea that social power and social dependence are tightly related concepts is not new, as it has been widely explored in social sciences and philosophy [6, 8] as well as in the area of multi-agent systems [17, 4]. According to [6] for instance, the *power* of an agent i over another agent j is based on the fact that j depends on i for the achievement of her goals. For example, Bill (who is the only mechanic in town), has power over Mary because Mary depends on Bill for achieving the goal of having her car repaired.

The input of the game-based power evaluations studied in this paper are boolean games [10], in which each player wants to achieve a certain goal represented by a propositional formula. The interesting aspect of boolean games is that they correspond to the specific subclass of normal-form games in which agents have binary preferences (i.e., payoffs can be either 0 or 1). Since the task of assessing the relative power of the agents in a boolean game is rather complex, the strategy we follow consists in decomposing it into two sub-tasks: construct and analyze *dependence functions* (that determine dependence relations in a boolean game) first, and then *power functions* (that evaluate power from dependencies).[1]

Our approach is axiomatic. We first establish postulates (or axioms) that dependence and power functions may satisfy. Such postulates can be seen as criteria useful to judge and compare different methods. Then, we analyze existing functions (e.g., Pagerank [12] and van den Brink & Gilles's power function [18]) on the basis of these criteria, and show that they fail to satisfy interesting postulates. So, we build new dependence and power functions and show that they satisfy such postulates. Finally, we show that game-based power evaluations can be obtained by combining dependence and power functions. The crucial point is that such combinations are *synergic*, in the sense that the combination of our two "local" methods (dependence functions and power functions) produces a "global" method (game-based power evaluation) that satisfies interesting properties.

The paper is organized as follows. Section 2 focuses on the problem of extracting a dependence relation from a boolean game by means of a dependence function. Section 3 is devoted to the axiomatic analysis of power functions. First some existing methods are critically investigated, then four new ones are proposed. Section 4 bridges the gap between

[1] The issue of extracting a dependence relation from a game has also been investigated by Bonzon et al. [5] for the class of boolean games (see Section 2 for more on this) and by Grossi & Turrini [9] for the entire class of normal-form games.

the first part and the second part of the paper. By means of the results obtained in Sections 2 and 3, we briefly show how certain structural properties of the boolean game determine specific properties of the power ranking of the agents.

2. DETERMINING DEPENDENCIES

We start by introducing the concept of boolean game as well as some related notions such as the concept of capability (Section 2.1). Then, we introduce the general concept of dependence function and propose a list of intuitively reasonable postulates that any dependence function is expected to satisfy (Section 2.2). Section 2.3 provides a critical discussion of an existing dependence function proposed by [5]: we show that it violates all postulates given in Section 2.2. In Section 2.4 a new dependence function based on the concept of veto player is proposed. We show that this dependence function satisfies all postulates of Section 2.2. Finally, in Section 2.5 we explore the relationship between this notion of dependence based on veto player and the concept of Nash equilibrium.

2.1 Boolean games and related notions

First of all, we shall introduce the concept of boolean game that has been proposed for the first time by [10] and further developed by [7].

Definition 1. A *boolean game* is a quadruple $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$, where:

- $\mathcal{A}$ is a non-empty finite set of agents (or players);
- Φ is a non-empty finite set of propositional atoms;
- γ_a is a formula of the language $\mathcal{L}_\Phi$ (i.e., the propositional language constructed on Φ) denoting the goal of agent a;
- $\Phi_a \subseteq \Phi$ is the set of all atoms controlled by agent a.

As standard [19], we assume that for all $a, b \in \mathcal{A}$, $\Phi_a \cap \Phi_b = \emptyset$ and that $\bigcup_{a \in \mathcal{A}} \Phi_a = \Phi$ (i.e., the controlled atoms form a partitition of the set Φ). Moreover, we assume that each γ_a is a satisfiable formula of $\mathcal{L}_\Phi$.

The following definition introduces the notion of strategy in a boolean game.

Definition 2. Given a boolean game $\mathbf{B}$ and a coalition $C \subseteq \mathcal{A}$, a *strategy* of coalition C is a function

$$s_C : \bigcup_{a \in C} \Phi_a \longrightarrow \{\top, \bot\}$$

which consists in setting the truth value of each atom controlled by some agent in C either to $\top$ (true) or $\bot$ (false). The set of all strategies of coalition C is denoted by S_C.

For notational convenience, strategies of the grand coalition $\mathcal{A}$ are denoted by $s, s', \ldots$ instead of $s_\mathcal{A}, s'_\mathcal{A}, \ldots$ and strategies for single-agent coalitions are denoted by $s_a, s_b, \ldots$ instead of $s_{\{a\}}, s_{\{b\}}, \ldots$ Moreover, we write S instead of $S_\mathcal{A}$ and S_a instead $S_{\{a\}}$.

In addition, we write $s \models \varphi$ to mean that the valuation defined by the strategy s satisfies formula $\varphi \in \mathcal{L}_\Phi$. We write $\models \varphi$ to mean that the propositional formula φ is valid in propositional logic, i.e., we have $s \models \varphi$ for all valuations s.

Finally, given a strategy s_C of the coalition C and a strategy $s'_{\mathcal{A} \setminus C}$ of the coalition $\mathcal{A} \setminus C$, we let $s_C \circ s'_{\mathcal{A} \setminus C}$ denote the strategy in S such that:

- (i) for all $p \in \bigcup_{a \in C} \Phi_a$, $s_C \circ s'_{\mathcal{A} \setminus C}(p) = s_C(p)$;
- (ii) for all $p \in \bigcup_{a \in \mathcal{A} \setminus C} \Phi_a$, $s_C \circ s'_{\mathcal{A} \setminus C}(p) = s'_{\mathcal{A} \setminus C}(p)$.

It is worth noting that, since each formula γ_a is satisfiable in propositional logic, all agents together have the capability of making an agent satisfied. That is, for every goal formula γ_a, there exists $s \in S$ such that $s \models \gamma_a$.

As a last concept for this section, we introduce the well-known concept of *capability* that is studied in the context of Coalition Logic (CL) [13] and Coalition Logic of Propositional Control (CL-PC) [19]. This is also called α-ability.

Definition 3. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ be a boolean game and let $\varphi \in \mathcal{L}_\Phi$. We say that coalition C has the *capability* of making formula φ true regardless of what the agents outside C do (or, C is an *effective coalition* for φ for short), denoted by $\mathtt{Can}_\mathbf{B}(C, \varphi)$, iff there exists $s_C \in S_C$ such that for all $s_{\mathcal{A} \setminus C} \in S_{\mathcal{A} \setminus C}$, $s_C \circ s_{\mathcal{A} \setminus C} \models \varphi$.

The following definition introduces the notion of minimal effective coalition for a given outcome φ that will be used in next section for our axiomatic analysis of dependence functions.

Definition 4. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ be a boolean game and let $\varphi \in \mathcal{L}_\Phi$. We say that coalition C is a *minimal coalition* having the capability of of making formula φ true regardless of what the agents outside C do (or, C is a *minimal effective coalition* for φ for short), denoted by $\mathtt{MinCan}_\mathbf{B}(C, \varphi)$, iff $\mathtt{Can}_\mathbf{B}(C, \varphi)$ holds and there is no $B \subset C$ such that $\mathtt{Can}_\mathbf{B}(B, \varphi)$ holds.

2.2 Postulates for dependence functions

A dependence function is a function that takes a boolean game as an input and transforms it into a dependence relation between the agents in the game. In particular:

Definition 5. A *dependence function* is a function $\mathbf{\Delta}$ transforming every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ into a binary relation $\mathcal{R}$ on $\mathcal{A}$, called a dependence relation.

Intuitively, $a\mathcal{R}b$ means that a depends on b, or equivalently that b has power over a.

Here we adopt an axiomatic approach to the analysis of dependence functions. Specifically, we propose four intuitively reasonable postulates for this family of functions. Our first postulate says the following: if agent a can achieve her goal γ_a regardless of what the other agents do then a does not depend on anybody, except perhaps on herself. More formally:

Postulate 1. A dependence function $\mathbf{\Delta}$ satisfies *sensitivity to agent capability* (Cap) iff for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ and $\forall a, b \in \mathcal{A}$, the following holds: if $\mathtt{Can}_\mathbf{B}(\{a\}, \gamma_a)$ and $a \neq b$ then $\langle a, b \rangle \notin \mathbf{\Delta}(\mathbf{B})$.

Our second postulate says the following: if b does not belong to any minimal effective coalition that can make a's goal satisfied then a does not depend on b.

Postulate 2. A dependence function $\mathbf{\Delta}$ satisfies *Absence of power* (AP) iff for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ and $\forall a, b \in \mathcal{A}$, the following holds: if there is no $C \subseteq \mathcal{A}$ such that $\mathtt{MinCan}_\mathbf{B}(C, \gamma_a)$ and $b \in C$ then $\langle a, b \rangle \notin \mathbf{\Delta}(\mathbf{B})$.

In order to formally state our third postulate we need to define first the following notion of agent dominance. The idea is that agent a dominates agent b with respect to agent

c if and only if, for every minimal effective coalition that can make c's goal satisfied, if b belongs to this coalition then a belongs to it as well.

Definition 6. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game and let $a, b, c \in \mathcal{A}$. We say that a dominates b with respect to c in $\mathbf{B}$, denoted by $\mathtt{Dom}_{\mathbf{B}}(a, b, c)$, iff $\forall C \subseteq \mathcal{A}$, if $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_c)$ and $b \in C$ then $a \in C$.

Our third postulate says that if agent a dominates agent b with respect to agent c then b has power over c only if b has power over c.

Postulate 3. A dependence function $\mathbf{\Delta}$ satisfies *Agent Dominance* (Dom) iff for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ and $\forall a, b, c \in \mathcal{A}$ the following holds: if $\mathtt{Dom}_{\mathbf{B}}(a, b, c)$ and $\langle c, b\rangle \in \mathbf{\Delta}(\mathbf{B})$ then $\langle c, a\rangle \in \mathbf{\Delta}(\mathbf{B})$.

Finally, our last postulate for dependence function says that if agent b's goal implies agent a's goal (i.e., b's goal is achieved only if a's goal is also achieved), then all agents who have power over a also have power over b.

Postulate 4. A dependence function $\mathbf{\Delta}$ satisfies *Implication* (Imp) iff for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ and $\forall a, b, c \in \mathcal{A}$, the following holds: if $\models \gamma_b \rightarrow \gamma_a$ and $\langle a, c\rangle \in \mathbf{\Delta}(\mathbf{B})$ then $\langle b, c\rangle \in \mathbf{\Delta}(\mathbf{B})$.

As a side note, we observe that *transitivity* does not seem to be a reasonable postulate for a notion of dependence based on boolean games. This is the reason why we do not consider it here. A counterexample to transitivity for dependence is the following. Suppose Mary depends on Bill, since Bill is the only mechanic in town and Mary has the goal of having her car repaired. Moreover, suppose Bill depends on Alice, since Alice is the only baker in town and Bill wants to buy bread. This does not imply that Mary depends on Alice, as it might be the case that Mary does not want to buy bread.

2.3 An existing dependence function

Bonzon et al. [5] have introduced a dependence function based on the concept of relevance. Let us call it *relevance-based function* and denote it by Rbf. This dependence function is also used by [15]. A propositional atom p is said to be relevant for a boolean formula φ if and only if there is no equivalent boolean formula ψ where p does not occur. Given a boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$, let $RV_{\mathbf{B}}(a)$ be the set of all variables $p \in \Phi$ that are relevant for a's goal γ_a and let $RA_{\mathbf{B}}(a)$ be the set of all agents $b \in \mathcal{A}$ such that b controls at least one variable that is relevant for a (i.e., there is $p \in \Phi_b$ such that $p \in RV_{\mathbf{B}}(a)$).

According to Bonzon et al.:

Definition 7. Agent a has a relevance-based dependence on agent b in the boolean game $\mathbf{B}$, i.e., $\langle a, b\rangle \in \mathtt{Rbf}(\mathbf{B})$, if and only if $b \in RA_{\mathbf{B}}(a)$.

As the following theorem highlights Rbf does not satisfy any postulate given in the previous Section 2.2:

THEOREM 1. Rbf *violates* (Cap), (AP), (Dom) *and* (Imp).

PROOF (SKETCH). We only prove that Rbf violates (Cap) by means of the following counterexample. Consider any boolean game $\mathbf{B}$ in which $\Phi_1 = \{p\}$, $\Phi_2 = \{q\}$ and $\gamma_1 = p \vee q$. Clearly, $2 \in RA_{\mathbf{B}}(1)$. Thus, $\langle 1, 2\rangle \in \mathtt{Rbf}(\mathbf{B})$. But, $\mathtt{Can}_{\mathbf{B}}(\{1\}, \gamma_1)$. □

In the next section we consider a new dependence function, called *veto-based dependence function*, which is stronger than Bonzon et al.'s relevance-based dependence function and which satisfies the four postulates given in Section 2.2.

2.4 A new dependence function

The *veto-based dependence function*, denoted by Vbf, is based on the concept of veto player as defined by [20] and, more generally, in the context of cooperative game theory [11].

According to Vbf, agent a depends on agent b, if the intervention of b is necessary to ensure that a will achieve her goal γ_a. Specifically, a depends on b if and only if the coalition $\mathcal{A} \setminus \{b\}$ is not capable of making a's goal true, independent of what agent b chooses. More formally:

Definition 8. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game. We say that *agent a has a veto-based dependence on agent b in* $\mathbf{B}$, or simply, *agent a depends on agent b in* $\mathbf{B}$, i.e., $\langle a, b\rangle \in \mathtt{Vbf}(\mathbf{B})$, iff there is no $s_{\mathcal{A}\setminus\{b\}} \in S_{\mathcal{A}\setminus\{b\}}$ such that for all $s_b \in S_b$, $s_{\mathcal{A}\setminus\{b\}} \circ s_b \models \gamma_a$.

The following theorem highlights that the four postulates of Section 2.2 are all sound with respect to Vbf.

THEOREM 2. Vbf *satisfies* (Cap), (AP), (Dom), *and* (Imp).

PROOF (SKETCH). We only provide the sketch of proof for (AP) and (Dom) as an example.

Suppose that $\langle a, b\rangle \in \mathtt{Vbf}(\mathbf{B})$. By the definitions of Vbf and of the notion of capability, we have that:

(A) $\langle a, b\rangle \in \mathtt{Vbf}(\mathbf{B})$ iff, for all $C \subseteq \mathcal{A}$, if $\mathtt{Can}_{\mathbf{B}}(C, \gamma_a)$ then $b \in C$.

Furthermore, the following property holds for all $a \in \mathcal{A}$:

(B) there exists $C \subseteq \mathcal{A}$ such that $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_a)$.

Thus, from the previous items (A) and (B), it follows that there exists $C \subseteq \mathcal{A}$ such that $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_a)$ and $b \in C$. This shows that Vbf satisfies (AP).

Let us prove that Vbf satisfies (Dom). Suppose that $\mathtt{Dom}_{\mathbf{B}}(a, b, c)$ and $\langle c, b\rangle \in \mathtt{Vbf}(\mathbf{B})$. As shown above, we have $\langle c, b\rangle \in \mathtt{Vbf}(\mathbf{B})$ iff for all $C \subseteq \mathcal{A}$, if $\mathtt{Can}_{\mathbf{B}}(C, \gamma_c)$ then $b \in C$. Thus, for all $C \subseteq \mathcal{A}$, if $\mathtt{Can}_{\mathbf{B}}(C, \gamma_c)$ then $b \in C$. Thus, since $\mathtt{Dom}_{\mathbf{B}}(a, b, c)$ holds, we have that for all $C \subseteq \mathcal{A}$, if $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_c)$ then $a \in C$. The following property holds:

(C) if (for all $C \subseteq \mathcal{A}$, if $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_c)$ then $a \in C$) then (for all $C \subseteq \mathcal{A}$, if $\mathtt{Can}_{\mathbf{B}}(C, \gamma_c)$ then $a \in C$).

Hence, we have that for all $C \subseteq \mathcal{A}$, if $\mathtt{Can}_{\mathbf{B}}(C, \gamma_c)$ then $a \in C$. By the previous item (A), this is equivalent to $\langle c, a\rangle \in \mathtt{Vbf}(\mathbf{B})$. This shows that Vbf satisfies (Dom). □

2.5 Dependencies and Nash equilibria

In this final section we explore the connection between Vbf and Nash equilibria.

Clearly boolean games correspond to a specific subclass of games in normal form, namely games in normal form with binary utility functions $u_a : S \longrightarrow \{0, 1\}$ defined by: (i) $u_a(s) = 0$ if $s \models \neg\gamma_a$, and (ii) $u_a(s) = 1$ if $s \models \gamma_a$. Pure-strategy Nash equilibria and strict Nash equilibria are defined exactly as in standard game theory.

Definition 9. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game. Moreover, let $s_a \in S_a$, $s_{\mathcal{A}\setminus\{a\}} \in S_{\mathcal{A}\setminus\{a\}}$ and $s = s_a \circ s_{\mathcal{A}\setminus\{a\}}$. Strategy $s \in S$ is a Nash equilibrium iff, for all $a \in \mathcal{A}$ and for all $s'_a \in S_a$, $u_a(s_a \circ s_{\mathcal{A}\setminus\{a\}}) \geq u_a(s'_a \circ s_{\mathcal{A}\setminus\{a\}})$. Strategy $s \in S$ is a *strict* Nash equilibrium iff, for all $a \in \mathcal{A}$ and for all $s'_a \in S_a \setminus \{s_a\}$, $u_a(s_a \circ s_{\mathcal{A}\setminus\{a\}}) > u_a(s'_a \circ s_{\mathcal{A}\setminus\{a\}})$.

The following definition captures the concept of k-resilient Nash equilibrium by Abraham et al. [1]:[2] a strategy profile s is said to be a k-resilient Nash equilibrium if no coalition of size at most k, taking the actions of its complements as given, can cooperatively deviate in a way that benefits *some of its members*. We also introduce the new concept of *strict* k-resilient Nash equilibrium.

Definition 10. Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game. Strategy $s \in S$ is a k-resilient Nash equilibrium iff for all $C \subseteq \mathcal{A}$ with $|C| \leq k$ and for all $s'_C \in S_C$, $u_a(s_C \circ s_{\mathcal{A}\setminus C}) \geq u_a(s'_C \circ s_{\mathcal{A}\setminus C})$ for all $a \in C$. Strategy $s \in S$ is a *strict* k-resilient Nash equilibrium iff for all $C \subseteq \mathcal{A}$ with $|C| \leq k$ and for all $s'_C \in S_C \setminus \{s_C\}$, $u_a(s_C \circ s_{\mathcal{A}\setminus C}) > u_a(s'_C \circ s_{\mathcal{A}\setminus C})$ for all $a \in C$.

Note that strategy s is a 1-resilient Nash equilibrium if and only if it is a Nash equilibrium, and it is a strict 1-resilient Nash equilibrium if and only if it is a strict Nash equilibrium. Furthermore, if $|\mathcal{A}| = k$ then strategy $s \in S$ is a *strict* k-resilient Nash equilibrium if and only if s is the unique strategy profile that makes all agents happy (i.e., $s \models \gamma_a$ for all $a \in \mathcal{A}$ and, $s' \models \neg\gamma_a$ for all $s' \neq s$ and for all $a \in \mathcal{A}$). This implies that, for every boolean game $\mathbf{B}$ with k players, there is at most one strict k-resilient Nash equilibrium.

Our first result concerning the link between `Vbf` and Nash equilibria is that the dependence relation extracted by means of `Vbf` from a k-player boolean game with one strict k-resilient Nash equilibrium and in which each agent controls at least one propositional atom is complete. More formally:

Theorem 3. *Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game with k players. If $\mathbf{B}$ has one strict k-resilient Nash equilibrium and $\Phi_a \neq \emptyset$ for all $a \in \mathcal{A}$, then* $\mathtt{Vbf}(\mathbf{B}) = \mathcal{A} \times \mathcal{A}$.

Proof (Sketch). As observed above, if $|\mathcal{A}| = k$ and s is a strict k-resilient Nash equilibrium then, s is the only strategy profile that makes all agents happy, while all other strategy profiles s' make all agents unhappy. Therefore, for all $a \in \mathcal{A}$, a can achieve his goal γ_a only if every agent b in $\mathcal{A}$ chooses the component s_b in s. Hence, for all $a \in \mathcal{A}$, a depends on all other agents. The condition $\Phi_a \neq \emptyset$ for all $a \in \mathcal{A}$ guarantees that every agent b in $\mathcal{A}$ has more than one choice available. Therefore, b can indeed deviate from the strategy profile s by playing something different from s_b and making all agents unhappy. $\square$

A more interesting result concerns the family of two-player boolean games with more than one strict Nash equilibrium. A typical example of games belonging to this family are pure coordination games [16] in which the two players in the game achieve their common goal only if they together play one of the many strict Nash equilibria. We can prove that every two-player boolean game with more than one strict Nash equilibrium induces a complete dependence relation. Intuitively, this means that in a two-player game with more than one strict Nash equilibrium (e.g., a two-player pure coordination game) the agents depend on each other.

[2]Abraham et al. uses the term '*strongly* k-resilient Nash equilibrium' but, in order simplify exposition, we simply call it 'k-resilient Nash equilibrium'.

	s_b^1	s_b^2	s_b^3	s_b^4
s_a^1	**1,1**	$*$,0	0,0	$*$,0
s_a^2	0,$*$	$*$,$*$	0,$*$	$*$,$*$
s_a^3	0,0	$*$,0	**1,1**	$*$,0
s_a^4	0,$*$	$*$,$*$	0,$*$	$*$,$*$

Figure 1: Boolean game with two strict Nash equilibria. Symbol $*$ means that the value is irrelevant.

Theorem 4. *Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a two-player boolean game. If $\mathbf{B}$ has more than one strict Nash equilibrium, then* $\mathtt{Vbf}(\mathbf{B}) = \mathcal{A} \times \mathcal{A}$.

Proof (Sketch). We give the general idea of the proof with the aid of Figure 1. Let $\mathbf{B}$ be any boolean game such that $\mathcal{A} = \{a, b\}$. Moreover, let $SN_{\mathbf{B}}$ be the set of strict Nash equilibria in $\mathbf{B}$ with $|SN_{\mathbf{B}}| > 1$. The first thing to observe is that two different strict Nash equilibria in $SN_{\mathbf{B}}$ necessarily differ in all their components: for all $s, s' \in SN_{\mathbf{B}}$, if $s \neq s'$ then $s_a \neq s'_a$ for all $a \in \mathcal{A}$. From this fact, one can prove that every row in the boolean game matrix contains at least one 0 for the row player (i.e., player a) and that every column in the boolean game matrix contains at least one 0 for the column player (i.e., player b). This is clearly shown by Figure 1 in which a boolean game with two strict Nash equilibria $s_a^1 \circ s_b^1$ and $s_a^3 \circ s_b^3$ is represented. Hence, it follows that $\mathtt{Vbf}(\mathbf{B}) = \{a, b\} \times \{a, b\}$, since there is no $s_a \in S_a$ such that for all $s_b \in S_b$, $s_a \circ s_b \models \gamma_a$, and there is no $s_b \in S_b$ such that for all $s_a \in S_a$, $s_b \circ s_a \models \gamma_b$. $\square$

Note that Theorem 4 cannot be generalized to one-player games. Indeed, as observed above, if a game has k players, then it has at most one strict k-resilient Nash equilibrium. Thus, a one-player game has at most one strict Nash equilibrium.

The following theorem provides a generalization of the preceding Theorem 4 to boolean games with k players that exploits the concept of strict k-resilient Nash equilibrium.

Theorem 5. *Let $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$ be a boolean game with k players. If $\mathbf{B}$ has more than one strict $(k-1)$-resilient Nash equilibrium, then* $\mathtt{Vbf}(\mathbf{B}) = \mathcal{A} \times \mathcal{A}$.

3. EVALUATING POWER

We turn to the second part of the paper, that is, the construction and analysis of power functions. The input of such a function is a dependence graph.

Definition 11. A *dependence graph* is an ordered pair $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$, where $\mathcal{A}$ is a finite set of agents and $\mathcal{R}$ a binary relation on $\mathcal{A}$, called a *dependence relation.*

The output of a power function is a *ranking* of the agents.

Definition 12. A *ranking* on a set of agents $\mathcal{A}$ is a total and transitive binary relation $\preceq$ on $\mathcal{A}$. Intuitively, $a \preceq b$ means that a is *at least as powerful as* b. So, $a \prec b$ (i.e., $b \not\preceq a$) means that a is *strictly more powerful than* b.

Note that $a \prec b$ can equivalently be read as a takes *precedence* over b with respect to power, or a is ranked *above* b with respect to power. So, it is natural to put the more powerful agent on the left-hand side of the $\prec$ symbol. Note also that $a \preceq b$ may be written as $\langle a, b\rangle \in \preceq$, if it is more convenient. Similarly, $a \prec b$ may be written as $\langle b, a\rangle \notin \preceq$.

We are ready to define the main concept of Section 3.

Definition 13. A *power function* $\mathbf{\Pi}$ transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ into a ranking on $\mathcal{A}$.

Note that such functions have been efficiently investigated in particular in [3], where it is essentially shown that no function can satisfy certain postulates all together.

3.1 Postulates for power functions

In the present section, we establish postulates (or axioms) for power functions. The first group of axioms captures the notion of *transitivity* studied in e.g. [3]. The principle of transitivity consists of two ideas: more dependents means more power; more powerful dependents means more power as well.

As a preliminary, we need a notation for the *dependents* of an agent a.

Definition 14. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ be a dependence graph and $a \in \mathcal{A}$. We denote by $\mathtt{Dep}_{\mathbf{D}}(a)$ the set of all *dependents* of a in $\mathbf{D}$, i.e., $\mathtt{Dep}_{\mathbf{D}}(a) = \{b \in \mathcal{A} \mid b\mathcal{R}a\}$.

Throughout Section 3, when the context is clear, we may omit certain subscripts, e.g., we may write $\mathtt{Dep}(a)$ for short.

Next, we need to construct 3 relations between groups of agents from a relation $\preceq$ between agents. More precisely, $\langle A, B\rangle \in \mathtt{G1}(\preceq)$ iff the agents of A are at least as numerous and powerful as those of B. The other relations G2 and G3 correspond to the same idea plus an advantage for A in cardinality or quality, respectively.

Definition 15. Let $\preceq$ be a ranking on a set $\mathcal{A}$. We denote by $\mathtt{G1}_{\mathcal{A}}(\preceq)$, $\mathtt{G2}_{\mathcal{A}}(\preceq)$, and $\mathtt{G3}_{\mathcal{A}}(\preceq)$ the binary relations on the power set of $\mathcal{A}$ such that $\forall\, A, B \subseteq \mathcal{A}$:

- $\langle A, B\rangle \in \mathtt{G1}(\preceq)$ iff $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ (i.e., there exists an injective function f from B to A) s.t. $\forall\, b \in B$, $f(b) \preceq b$;
- $\langle A, B\rangle \in \mathtt{G2}(\preceq)$ iff $|B| < |A|$ and $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$;
- $\langle A, B\rangle \in \mathtt{G3}(\preceq)$ iff $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$ and $\exists\, b \in B$, $f(b) \prec b$.

We are ready to define the axioms of transitivity.

Postulate 5. A power function $\mathbf{\Pi}$ satisfies the *first*, *second*, or *third form of transitivity*, denoted by (T1), (T2), or (T3), respectively, iff for every dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$, $\forall\, a, b \in \mathcal{A}$:

(T1) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{G1}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle a, b\rangle \in \mathbf{\Pi}(\mathbf{D})$;
(T2) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{G2}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle b, a\rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(T3) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{G3}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle b, a\rangle \notin \mathbf{\Pi}(\mathbf{D})$.

We turn to the second group of axioms, which captures the notion of *transitivity with dilution* studied in e.g. [3]. By dilution, we mean that the power a gives to certain agents has to be *evenly shared* between them.

Again, we need preliminaries. First, we need a notation for the agents having power over a:

Definition 16. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ be a dependence graph and $a \in \mathcal{A}$. We denote by $\mathtt{Pow}_{\mathbf{D}}(a)$ the *set of all agents having power over a in* $\mathbf{D}$, i.e., $\mathtt{Pow}_{\mathbf{D}}(a) = \{b \in \mathcal{A} \mid a\mathcal{R}b\}$.

Next, we need four relations between groups of agents. Intuitively, $\langle A, B\rangle \in \mathtt{GD1}(\preceq)$ iff the agents of A are at least as numerous and powerful as those of B and they do not depend on more agents, i.e., there is no more power dilution. The other relations GD2, GD3, and GD4 capture the same idea plus an advantage for A in cardinality, quality, or dilution, respectively.

Definition 17. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ be a dependence graph and $\preceq$ a ranking on $\mathcal{A}$. We denote by $\mathtt{GD1}_{\mathbf{D}}(\preceq)$, $\mathtt{GD2}_{\mathbf{D}}(\preceq)$, $\mathtt{GD3}_{\mathbf{D}}(\preceq)$, and $\mathtt{GD4}_{\mathbf{D}}(\preceq)$ the binary relations on the power set of $\mathcal{A}$ such that $\forall\, A, B \subseteq \mathcal{A}$:

- $\langle A, B\rangle \in \mathtt{GD1}(\preceq)$ iff $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$ and $\forall\, b \in B$, $|\mathtt{Pow}[f(b)]| \leq |\mathtt{Pow}(b)|$;
- $\langle A, B\rangle \in \mathtt{GD2}(\preceq)$ iff $|B| < |A|$ and $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$ and $\forall\, b \in B$, $|\mathtt{Pow}[f(b)]| \leq |\mathtt{Pow}(b)|$;
- $\langle A, B\rangle \in \mathtt{GD3}(\preceq)$ iff $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$, $\forall\, b \in B$, $|\mathtt{Pow}[f(b)]| \leq |\mathtt{Pow}(b)|$, and $\exists\, b \in B$, $f(b) \prec b$;
- $\langle A, B\rangle \in \mathtt{GD4}(\preceq)$ iff $\exists\, f : B \xrightarrow{\mathtt{inj}} A$ s.t. $\forall\, b \in B$, $f(b) \preceq b$, $\forall\, b \in B$, $|\mathtt{Pow}[f(b)]| \leq |\mathtt{Pow}(b)|$, and $\exists b \in B$, $|\mathtt{Pow}[f(b)]| < |\mathtt{Pow}(b)|$.

We can define the axioms of transitivity with dilution.

Postulate 6. A power function $\mathbf{\Pi}$ satisfies the *first*, *second*, *third*, or *fourth form of transitivity with dilution*, denoted by (TD1), (TD2), (TD3), or (TD4), respectively, iff for every dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$, $\forall\, a, b \in \mathcal{A}$:

(TD1) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{GD1}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle a, b\rangle \in \mathbf{\Pi}(\mathbf{D})$;
(TD2) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{GD2}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle b, a\rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(TD3) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{GD3}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle b, a\rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(TD4) if $\langle \mathtt{Dep}(a), \mathtt{Dep}(b)\rangle \in \mathtt{GD4}[\mathbf{\Pi}(\mathbf{D})]$, then $\langle b, a\rangle \notin \mathbf{\Pi}(\mathbf{D})$.

Finally, we introduce a last group of axioms capturing the notion of *transitivity with redundancy*. Intuitively, there is redundancy in the power of an agent a iff a depends on herself (possibly indirectly) or two direct dependents of a *overlap*, i.e., at least one depends on the other (possibly indirectly) or they have (possibly indirect) dependents in common. As far as we know, this group of axioms is new.

First, we need to define the *extended dependents* of a, i.e., the agents depending, directly or indirectly, on a.

Definition 18. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ be a dependence graph. We denote by $\mathtt{Exat}_{\mathbf{D}}$ the function on $\{1, 2, \ldots\} \times \mathcal{A}$ such that $\forall\, i \in \{1, 2, \ldots\}$, $\forall\, a \in \mathcal{A}$, $\mathtt{Exat}_{\mathbf{D}}(i, a)$ is the *set of all extended dependents of a in* $\mathbf{D}$ *at distance* i, i.e.,

$$\mathtt{Exat}(i, a) = \begin{cases} \mathtt{Dep}(a) & \text{if } i = 1; \\ \bigcup_{b \in \mathtt{Exat}(i-1, a)} \mathtt{Dep}(b) & \text{if } 1 < i. \end{cases}$$

We define that $\mathtt{Ext}_{\mathbf{D}}(a) = \bigcup_{i \in \{1,2,\ldots\}} \mathtt{Exat}_{\mathbf{D}}(i, a)$.

Next, we need to formalize the notion of overlapping agents:

Definition 19. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R}\rangle$ be a dependence graph and $A \subseteq \mathcal{A}$. We denote by $\mathtt{Ovr}_{\mathbf{D}}(A)$ the fact that *two agents of A overlap in* $\mathbf{D}$, i.e., $\exists\, a, b \in A$, $a \neq b$ and ($a \in \mathtt{Ext}(b)$ or $b \in \mathtt{Ext}(a)$ or $\mathtt{Ext}(a) \cap \mathtt{Ext}(b) \neq \emptyset$).

As a last preliminary, we need four relations between groups of agents to represent equivalence or advantage in cardinality, quality, and redundancy.

Definition 20. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph and $\preceq$ a ranking on $\mathcal{A}$. We denote by $\texttt{GR1}_{\mathbf{D}}(\preceq)$, $\texttt{GR2}_{\mathbf{D}}(\preceq)$, $\texttt{GR3}_{\mathbf{D}}(\preceq)$, and $\texttt{GR4}_{\mathbf{D}}(\preceq)$ the binary relations on the power set of $\mathcal{A}$ such that $\forall\, A, B \subseteq \mathcal{A}$:

- $\langle A, B \rangle \in \texttt{GR1}(\preceq)$ iff $\neg\texttt{Ovr}(A)$ and $\exists\, f : B \xrightarrow{\texttt{inj}} A$ s.t. $\forall\, b \in B,\ f(b) \preceq b$;
- $\langle A, B \rangle \in \texttt{GR2}(\preceq)$ iff $|B| < |A|$, $\neg\texttt{Ovr}(A)$, and $\exists\, f : B \xrightarrow{\texttt{inj}} A$ s.t. $\forall\, b \in B,\ f(b) \preceq b$;
- $\langle A, B \rangle \in \texttt{GR3}(\preceq)$ iff $\neg\texttt{Ovr}(A)$ and $\exists\, f : B \xrightarrow{\texttt{inj}} A$ s.t. $\forall\, b \in B,\ f(b) \preceq b$ and $\exists\, b \in B,\ f(b) \prec b$;
- $\langle A, B \rangle \in \texttt{GR4}(\preceq)$ iff $\neg\texttt{Ovr}(A)$, $\texttt{Ovr}(B)$, and $\exists\, f : B \xrightarrow{\texttt{inj}} A$ s.t. $\forall\, b \in B,\ f(b) \preceq b$.

We can define the axioms of transitivity with redundancy.

Postulate 7. A power function $\mathbf{\Pi}$ satisfies the *first*, *second*, *third*, *fourth*, or *fifth form of transitivity with redundancy*, denoted by (TR1), (TR2), (TR3), (TR4), or (TR5), respectively, iff for every dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$, $\forall a, b \in \mathcal{A}$:

(TR1) if $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GR1}[\mathbf{\Pi}(\mathbf{D})]$ and $a \notin \texttt{Ext}(a)$, then $\langle a, b \rangle \in \mathbf{\Pi}(\mathbf{D})$;
(TR2) if $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GR2}[\mathbf{\Pi}(\mathbf{D})]$ and $a \notin \texttt{Ext}(a)$, then $\langle b, a \rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(TR3) if $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GR3}[\mathbf{\Pi}(\mathbf{D})]$ and $a \notin \texttt{Ext}(a)$, then $\langle b, a \rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(TR4) if $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GR4}[\mathbf{\Pi}(\mathbf{D})]$ and $a \notin \texttt{Ext}(a)$, then $\langle b, a \rangle \notin \mathbf{\Pi}(\mathbf{D})$;
(TR5) if $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GR1}[\mathbf{\Pi}(\mathbf{D})]$, $a \notin \texttt{Ext}(a)$, and $b \in \texttt{Ext}(b)$, then $\langle b, a \rangle \notin \mathbf{\Pi}(\mathbf{D})$.

We conclude this section with dependencies and incompatibilities between the twelve postulates we have defined.

THEOREM 6. *The following dependencies hold:*

- (T1) *implies* (TD1) *and* (TR1)*;*
- (T2) *implies* (TD2) *and* (TR2)*;*
- (T3) *implies* (TD3) *and* (TR3)*.*

PROOF (SKETCH). The axiom (T1) implies the axiom (TD1), because the antecedent $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{GD1}[\mathbf{\Pi}(\mathbf{D})]$ implies the antecedent $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{G1}[\mathbf{\Pi}(\mathbf{D})]$. A similar reasoning holds for the other implications. □

THEOREM 7. *The six following combinations of postulates are unsatisfiable:* (T1) + (TD4)*;* (T1) + (TR4)*;* (T1) + (TR5)*;* (TD1) + (TR4)*;* (TD1) + (TR5)*; and* (TR1) + (TD4)*.*

PROOF (SKETCH). Suppose $\mathbf{\Pi}$ satisfies both (T1) and (TD4). Let $\mathcal{R} = \{ca, ca', db\}$. Then, by (T1), $\langle a, b \rangle \in \mathbf{\Pi}(\mathbf{D})$. But, by (TD4), $\langle a, b \rangle \notin \mathbf{\Pi}(\mathbf{D})$, impossible. Similar examples can be found for the other incompatibilities. □

3.2 Three existing power functions

This section examines three existing power functions studied in the literature. The first power function we consider is the *score-based function* axiomatized by Rubinstein in [14].

Definition 21. The power function $\texttt{Sbf}$ transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking on $\mathcal{A}$ such that $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \texttt{Sbf}(\mathbf{D})$ iff $|\texttt{Dep}(b)| \leq |\texttt{Dep}(a)|$.

We analyze $\texttt{Sbf}$ through the postulates of Section 3.1.

THEOREM 8. $\texttt{Sbf}$ *satisfies* (T1)*,* (T2)*,* (TD1)*,* (TD2)*,* (TR1)*,* (TR2)*, but violates* (T3)*,* (TD3)*,* (TD4)*,* (TR3)*-*(TR5)*.*

PROOF (SKETCH). The axiom (T1) holds, because the antecedent $\langle \texttt{Dep}(a), \texttt{Dep}(b) \rangle \in \texttt{G1}[\texttt{Sbf}(\mathbf{D})]$ entails $|\texttt{Dep}(b)| \leq |\texttt{Dep}(a)|$. The case of (T2) is similar. The axioms (TD4), (TR4), and (TR5) are violated, because, with e.g. $\mathcal{R} = \{a'a, a''a', b'b\}$, they all imply $\langle b, a \rangle \notin \texttt{Sbf}(\mathbf{D})$. Theorems 6 and 7 deal with the other axioms. □

Another power function studied in the literature is the *van den Brink-Gilles function* [18], which consists in adding power dilution in $\texttt{Sbf}$.

Definition 22. The power function $\texttt{BGf}$ transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking on $\mathcal{A}$ such that $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \texttt{BGf}(\mathbf{D})$ iff $\sum_{c \in \texttt{Dep}(b)} 1/|\texttt{Pow}(c)| \leq \sum_{c \in \texttt{Dep}(a)} 1/|\texttt{Pow}(c)|$.

Note that, in Theorem 4.2 of [18], it is shown that $\texttt{BGf}$ is equivalent to the well-known *Shapley power function*, if the "worth" of a coalition is naturally defined as the number of dependents of that coalition. A study of the Shapley function can be found in e.g. [2].

We turn to the postulate-based analysis of $\texttt{BGf}$.

THEOREM 9. $\texttt{BGf}$ *satisfies* (TD1)*,* (TD2)*,* (TD4)*, but violates* (T1)*-*(T3)*,* (TD3)*,* (TR1)*-*(TR5)*.*

PROOF (SKETCH). The satisfaction part is trivial. With $\mathcal{R} = \{ca, ca', da, da', eb\}$, both (T2) and (TR2) imply $\langle b, a \rangle \notin \texttt{BGf}(\mathbf{D})$, which contradicts $\texttt{BGf}$. With $\mathcal{R} = \{a'a, a''a', b'b\}$, (T3), (TD3), and (TR3) all imply $\langle b, a \rangle \notin \texttt{BGf}(\mathbf{D})$, impossible. Theorem 7 deals with the remaining postulates. □

We turn to *Pagerank* [12], which is an important and more recent power function, as well as one of the main constituents of the Google search engine. In Pagerank, every agent a transfers credit to every agent b. The main idea is the following: if a depends on b, then more credit is transferred. Such an increase is determined by a fixed *damping factor* δ.

Definition 23. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph, $a, b \in \mathcal{A}$, and $\delta \in (0, 1)$. We denote by $\texttt{Per}_{\mathbf{D}\delta}(a, b)$ the *percentage of credit transferred from* a *to* b *in* $\mathbf{D}$ *with damping* δ, i.e.:

$$\texttt{Per}(a, b) = \begin{cases} 1/|\mathcal{A}| & \text{if } \texttt{Pow}(a) = \emptyset; \\ (1-\delta)/|\mathcal{A}| & \text{if } b \notin \texttt{Pow}(a) \neq \emptyset; \\ \delta/|\texttt{Pow}(a)| + (1-\delta)/|\mathcal{A}| & \text{if } b \in \texttt{Pow}(a). \end{cases}$$

Definition 24. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph and let $\delta \in (0, 1)$. We denote by $\texttt{Cre}_{\mathbf{D}\delta}$ the function on $\{0, 1, \ldots\} \times \mathcal{A}$ such that $\forall i \in \{0, 1, \ldots\}$, $\forall a \in \mathcal{A}$, $\texttt{Cre}_{\mathbf{D}\delta}(i, a)$ is the *quantity of credit* a *possesses in* $\mathbf{D}$ *in step* i *with damping* δ, i.e.:

$$\texttt{Cre}(i, a) = \begin{cases} 1/|\mathcal{A}| & \text{if } i = 0; \\ \sum_{b \in \mathcal{A}} \texttt{Per}(b, a)\texttt{Cre}(i-1, b) & \text{otherwise.} \end{cases}$$

Definition 25. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph, $a \in \mathcal{A}$, and $\delta \in (0, 1)$. We denote by $\texttt{Pr}_{\mathbf{D}\delta}(a)$ the *final quantity of credit* a *possesses in* $\mathbf{D}$ *with damping* δ, i.e.:

$$\texttt{Pr}(a) = \lim_{i \to \infty} \texttt{Cre}(i, a).$$

Note that a characterisation of the final quantities of credit in Pagerank can be found in e.g. [12].

THEOREM 10. *Let* $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ *be a dependence graph and* $\delta \in (0,1)$*. The function* $a \mapsto \mathtt{Pr}(a)$ *is the unique function* w *from* $\mathcal{A}$ *to* $[0,1]$ *satisfying the two following points:*
(Sum) $\forall\, a \in \mathcal{A},\ w(a) = \sum_{b \in \mathcal{A}} \mathtt{Per}(b,a) w(b)$*;*
(Norm) $\sum_{a \in \mathcal{A}} w(a) = 1$.

We are ready to define the Pagerank power function.

Definition 26. Let $\delta \in (0,1)$. The power function $\mathtt{Prf}_\delta$ transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking s.t. $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \mathtt{Prf}_\delta(\mathbf{D})$ iff $\mathtt{Pr}_{\mathbf{D}\delta}(b) \leq \mathtt{Pr}_{\mathbf{D}\delta}(a)$.

We axiomatically analyze the Pagerank power function:

THEOREM 11. *Let* $\delta \in (0,1)$*. The power function* $\mathtt{Prf}_\delta$ *satisfies* (TD1)-(TD4)*, but violates* (T1)-(T3)*,* (TR1)-(TR5)*.*

PROOF (SKETCH). The satisfaction part follows from (Sum) and (Norm). The violation part holds, either because of Theorem 7 or because a disadvantage in dilution can overcome any advantage in cardinality or quality. □

To summarize, there are three families of axioms capturing the notions of transitivity, transitivity-with-dilution, and transitivity-with-redundancy, respectively. The function Sbf satisfies an even part of each family, BGf is more oriented to transitivity-with-dilution, and Prf fully satisfies it.

3.3 Three new power functions

In the present section, we construct three new power functions fully satisfying transitivity, transitivity-with-dilution, and transitivity-with-redundancy, respectively. Those three families of axioms have been defined in Section 3.1.

Our first power function is based on the three following ideas: the agents have *accounts* on which *points* can be added or removed; the more the dependents of a are numerous and have points on their accounts, the more a has points as well; the more rapidly an agent gets a great number of points, the higher she is ranked.

Definition 27. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph. We denote by $\mathtt{Ac}_\mathbf{D}$ the function on $\{0, 1, \ldots\} \times \mathcal{A}$ such that $\forall\, i \in \{0, 1, \ldots\}$, $\forall\, a \in \mathcal{A}$, $\mathtt{Ac}_\mathbf{D}(i,a)$ is the *account of* a *in* $\mathbf{D}$ *in the step* i, i.e.:

$$\mathtt{Ac}(i,a) = \begin{cases} 1 & \text{if } i = 0; \\ 1 + \Sigma_{b \in \mathtt{Dep}(a)} \mathtt{Ac}(i-1, b) & \text{otherwise.} \end{cases}$$

Definition 28. The power function Abf transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking s.t. $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \mathtt{Abf}(\mathbf{D})$ iff one of the two following cases holds:
- $\forall\, i \geq 0$, $\mathtt{Ac}(i,a) = \mathtt{Ac}(i,b)$;
- $\exists i \geq 0$, $\mathtt{Ac}(i,b) < \mathtt{Ac}(i,a)$ and $\forall j < i$, $\mathtt{Ac}(j,a) = \mathtt{Ac}(j,b)$.

We are ready to axiomatically analyze Abf.

THEOREM 12. Abf *satisfies* (T1)-(T3)*,* (TD1)-(TD3)*,* (TR1)-(TR3)*, but violates* (TD4)*,* (TR4)*,* (TR5)*.*

PROOF (SKETCH). We begin with (T1). Case A: the dependents of b have the same accounts as the corresponding dependents of a, in any step. Then, the account of a is at least as big as that of b, in any step. Case B: on the contrary, there is a difference, in some step. Let i be the smallest step where there is a difference. Then, the account of a is greater than that of b in step $i+1$ (and at least as big below $i+1$). The axiom (T3) holds, because of the same proof, except that Case A is impossible. Concerning (T2), it follows from the top priority of Step 0. The other axioms are satisfied or violated because of Theorems 6 and 7. □

Our second power function consists in introducing the notion of power dilution in Abf.

Definition 29. Let $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ be a dependence graph. We denote by $\mathtt{Dac}_\mathbf{D}$ the function on $\{0, 1, \ldots\} \times \mathcal{A}$ such that $\forall\, i \in \{0, 1, \ldots\}$, $\forall\, a \in \mathcal{A}$, $\mathtt{Dac}_\mathbf{D}(i,a)$ is the *dilution-based account of* a *in* $\mathbf{D}$ *in the step* i, i.e.:

$$\mathtt{Dac}(i,a) = \begin{cases} 1 & \text{if } i = 0; \\ 1 + \Sigma_{b \in \mathtt{Dep}(a)} (\mathtt{Dac}(i-1, b) / |\mathtt{Pow}(b)|) & \text{if } i > 0. \end{cases}$$

Definition 30. The power function Daf transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking s.t. $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \mathtt{Daf}(\mathbf{D})$ iff one of the two following cases holds:
- $\forall\, i \geq 0$, $\mathtt{Dac}(i,a) = \mathtt{Dac}(i,b)$;
- $\exists\, i \geq 0$, $\mathtt{Dac}(i,b) < \mathtt{Dac}(i,a)$ and $\forall\, j < i$, $\mathtt{Dac}(j,a) = \mathtt{Dac}(j,b)$.

We axiomatically analyze Daf.

THEOREM 13. Daf *satisfies* (TD1)-(TD4)*, but violates* (T1)-(T3)*,* (TR1)-(TR5)*.*

PROOF (SKETCH). The axiom (TD4) follows from the top priority of Step 0. Concerning the other satisfied postulates, the reasoning is similar to the one for Abf. The remaining postulates are violated, either because of Theorem 7 or because a disadvantage in dilution can overcome any advantage in cardinality or quality. □

Finally, we define a third power function by *counting* the extended supporters of the agents.

Definition 31. The power function Cbf transforms any dependence graph $\mathbf{D} = \langle \mathcal{A}, \mathcal{R} \rangle$ into the ranking such that $\forall\, a, b \in \mathcal{A}$, $\langle a, b \rangle \in \mathtt{Cbf}(\mathbf{D})$ iff $|\{b\} \cup \mathtt{Ext}(b)| \leq |\{a\} \cup \mathtt{Ext}(a)|$.

We analyze Cbf with the axioms of Section 3.2.

THEOREM 14. Cbf *satisfies* (TR1)-(TR5)*, but violates* (T1)-(T3)*,* (TD1)-(TD4)*.*

PROOF (SKETCH). The satisfaction part essentially follows from the following fact: if there is no redundancy in the power of an agent a, then $|\{a\} \cup \mathtt{Ext}(a)| = 1 + |\mathtt{Ext}(a)| = 1 + \sum_{b \in \mathtt{Dep}(a)} |\{b\} \cup \mathtt{Ext}(b)|$. The violation part holds, either because of Theorem 7 or because, in certain examples, a disadvantage w.r.t. redundancy can overcome any advantage w.r.t. cardinality or quality. □

4. COMBINING FUNCTIONS

As emphasized in the introduction, dependence functions and power functions can be combined to produce game-based power evaluations that satisfy interesting properties.

Definition 32. A *game-based power evaluation* is a function Γ transforming any boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a \in \mathcal{A}}, \{\Phi_a\}_{a \in \mathcal{A}} \rangle$ into a ranking on $\mathcal{A}$.

A first postulate for game-based power evaluation is the following: if a dominates b with respect to every agent in the game, then a should be at least as powerful as b. More formally:

Postulate 8. A game-based power evaluation Γ satisfies *Universal Dominance* (UDom) iff, for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$, $\forall\, a, b \in \mathcal{A}$, if $\forall\, c \in \mathcal{A}$, $\mathtt{Dom}_{\mathbf{B}}(a, b, c)$, then $\langle a, b\rangle \in \Gamma(\mathbf{B})$.

The second postulate says the following: if an agent a does not appear in any minimal effective coalition for the goal of a player in the game, then each agent in the game is at least as powerful as a. More formally:

Postulate 9. A game-based power evaluation Γ satisfies *Universal Absence of Power* (UAP) iff, for every boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$, $\forall\, a \in \mathcal{A}$, if there is no $C \subseteq \mathcal{A}$ and $b \in \mathcal{A}$ such that $\mathtt{MinCan}_{\mathbf{B}}(C, \gamma_b)$ and $a \in C$, then $\forall\, c \in \mathcal{A}$, $\langle c, a\rangle \in \Gamma(\mathbf{B})$.

Now, the idea is to combine dependence and power functions, in order to construct game-based power evaluations satisfying our postulates.

Definition 33. Let Δ be a dependence function and Π a power function. We denote by $\Pi \circ \Delta$ the *combination of* Π *and* Δ, i.e. the game-based power evaluation such that, for any boolean game $\mathbf{B} = \langle \mathcal{A}, \Phi, \{\gamma_a\}_{a\in\mathcal{A}}, \{\Phi_a\}_{a\in\mathcal{A}}\rangle$, $\Pi \circ \Delta(\mathbf{B}) = \Pi\langle \mathcal{A}, \Delta(\mathbf{B})\rangle$.

THEOREM 15. *Let* Δ *be a dependence function and* Π *a power function. If* Δ *satisfies* (Dom) *and* Π *satisfies* (T1) *or* (TD1)*, then* $\Pi \circ \Delta$ *satisfies* (UDom).

THEOREM 16. *Let* Δ *be a dependence function and* Π *a power function. If* Δ *satisfies* (AP) *and* Π *satisfies* (T1) *or* (TD1)*, then* $\Pi \circ \Delta$ *satisfies* (UAP).

5. CONCLUSION

Let's take stock. The main contributions of the paper are the following:

- a postulate-based analysis of the new dependence function Vbf and of the existing dependence function Rbf, with special emphasis on the advantages of the former compared to the latter;
- a descriptive analysis of the relationship between Vbf and the concepts of Nash equilibrium and k-resilient Nash equilibrium;
- a postulate-based analysis of three existing power functions Sbf, BGf, and Prf and four new power functions Abf, Daf, Cbf, and Dbf, with special emphasis on the fact that most of our new functions satisfy postulates that are violated by the existing ones.

As future work, we plan to characterize our dependence and power functions, that is, to find enough postulates so that only one method satisfies them all together. We also plan to provide an axiomatic analysis of a weaker notion of dependence, which consists in generalizing the concept of veto to coalitions. The idea is that an agent a depends on another agent b if and only if, there exists a coalition of which b is a member such that its intervention is necessary to ensure that a will achieve her goal.

6. REFERENCES

[1] I. Abraham, D. Dolev, and J. Y. Halpern. Lower bounds on implementing robust and resilient mediators. In *Proc. of the 5th conference on Theory of cryptography (TCC 2008)*, pages 302–319. 2008.

[2] T. Ågotnes, W. van der Hoek, M. Tennenholtz, and M. Wooldridge. Power in normative systems. In *Proc. of AAMAS 2009*, pages 145–152, 2009.

[3] A. Altman and M. Tennenholtz. Axiomatic Foundations for Ranking Systems. *Journal of Artifical Intelligence Research*, 31:473–495, 2008.

[4] G. Boella, L. Sauro, and L. van der Torre. From social power to social importance. *Web Intelligence and Agent Systems*, 5(4):393–404, 2007.

[5] E. Bonzon, M.-C. Lagasquie-Schiex, and J. Lang. Dependencies between players in boolean games. *Journal of Approximate Reasoning*, 50(6):899–914, 2009.

[6] C. Castelfranchi. The micro-macro constitution of power. *Protosociology*, 23:208–269, 2003.

[7] P. E. Dunne, S. Kraus, W. van der Hoek, and M. Wooldridge. Cooperative boolean games. In *Proc. of AAMAS 2008*, pages 1015–1022. 2008.

[8] A. I. Goldman. Toward a theory of social power. *Philosophical studies*, 23:221–268, 1972.

[9] D. Grossi and P. Turrini. Dependence in games and dependence games. *Autonomous Agents and Multi-Agent Systems*, 25(2):284–312, 2012.

[10] P. Harrenstein, W. van der Hoek, J.-J. C. Meyer, and C. Witteveen. Boolean games. In *Proc. of TARK VIII*, pages 287–298, 2001.

[11] M. Osborne and A. Rubinstein. *A course in game theory.* MIT Press, 1994.

[12] L. Page, S. Brin, R. Motwani, and T. Winograd. The PageRank Citation Ranking: Bringing Order to the Web. Technical report, Stanford Digital Library Technologies Project, 1998.

[13] M. Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 12(1):149–166, 2002.

[14] A. Rubinstein. Ranking the Participants in a Tournament. *SIAM Journal of Applied Mathematics*, 38:108–111, 1980.

[15] L. Sauro and S. Villata. Dependency in cooperative boolean games. *Journal of Logic and Computation*, 23(2):425–444, 2013.

[16] T. Schelling. *The strategy of conflict.* Harvard University Press, 1960.

[17] J. S. Sichman, R. Conte, C. Castelfranchi, and Y. Demazeau. A social reasoning mechanism based on dependence networks. In *Proc. of ECAI'94*, pages 188–192, 1994.

[18] R. van den Brink and R. P. Gilles. Measuring domination in directed graphs. *Social Networks*, 22(2):141–157, 2000.

[19] W. van der Hoek and M. Wooldridge. On the logic of cooperation and propositional control. *Artificial Intelligence*, 164(1-2):81–119, 2005.

[20] M. Wooldridge and P. E. Dunne. On the computational complexity of qualitative coalitional games. *Artificial Intelligence*, 158(1):27–73, 2004.